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MEDIA WITH EQUATIONS OF STATE THAT DEPEND ON DERIVATIVES

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1. Introduction. S. K. Godunov [1] introduced a wide class of quasilinear equations of the form

$$\frac{\partial}{\partial x^\alpha}\Big(\frac{\partial L^\alpha}{\partial q^\beta}\Big) = 0 \quad (\alpha = 0,\dots,n, \beta = 1,\dots,m), \quad L^\alpha = L^\alpha(q^1,\dots,q^m). \tag{1.1}$$

Here and below summation is carried out over repeated indices.

This class of equations possesses the following remarkable features: first, it admits the additional conservation law

$$\frac{\partial \Phi^\alpha}{\partial x^\alpha} = 0, \qquad \Phi^\alpha \equiv q^\beta \frac{\partial L^\alpha}{\partial q^\beta} - L^\alpha; \tag{1.2}$$

second, system (1.1) is symmetric, i.e., is written as

$$A^\alpha_{\beta\gamma}\frac{\partial q^\gamma}{\partial x^\alpha} = 0, \qquad A^\alpha_{\beta\gamma} = A^\alpha_{\gamma\beta} = \frac{\partial^2 L^\alpha}{\partial q^\beta \partial q^\gamma}. \tag{1.3}$$

The converse statement is also true. If a system of m conservation laws admits the additional conservation law, it is written as (1.1). Examples of the systems of equations of the form (1.1)–(1.3) can be found in [1–7]. This class of equations is appropriate for describing quasilinear hyperbolic equations modeling wave propagation in continua. Conservation law (1.2) is the energy (or entropy) conservation law for the media. A natural desire arises to extend this class so that it could be used to describe nonlinear waves in media with dispersion.

By a system with dispersion is meant a system of nonlinear equations having the form of exact conservation laws, i.e., written in the form

$$\frac{\partial}{\partial x^\alpha}\varphi^\alpha_\beta = 0 \qquad (\alpha = 0,\dots,n, \qquad \beta = 1,\dots,m),$$

where the functions φ^α_β depend on a finite number of derivatives with respect to independent variables (below we consider only the case where φ^α_β depends on q^β and their first derivatives) and having exact laws for energy and entropy conservation, if the notion of entropy makes sense in the system.

In this work this class is described in terms of variational derivatives, and numerous examples of mathematical models falling within this class are presented. In addition, the peculiarities of the media for which the equation of state depends on derivatives are analyzed. It appears that the most important thermodynamic relationships for them are also formulated in terms of variational derivatives.

2. Systems with Dispersion. Let us introduce the independent variables (x^α) $(\alpha = 0,\dots,n)$, dependent variables q^β $(\beta = 1,\dots,m)$, derivatives $q^\beta_\gamma \equiv \partial q^\beta/\partial x^\gamma$, $q^\beta_{\gamma\delta} \equiv \partial^2 q^\beta/\partial x^\gamma \partial x^\delta$, and functions $L^\alpha(q^\beta, q^\beta_\gamma)$. We denote the variational derivative of L^α as

$$\frac{\delta L^\alpha}{\delta q^\beta} \equiv \frac{\partial L^\alpha}{\partial q^\beta} - \frac{\partial}{\partial x^\gamma}\Big(\frac{\partial L^\alpha}{\partial q^\beta_\gamma}\Big).$$

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Definition. The system of equations of the form

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\delta L^\alpha}{\delta q^\beta}\right) = 0 \tag{2.1}$$

is called a generalization of Godunov's system.

The following statement is true.

Theorem. *System* (2.1) *admits the conservation law*

$$\frac{\partial \Phi^\alpha}{\partial x^\alpha} = 0, \qquad \Phi^\alpha \equiv q^\beta \frac{\delta L^\alpha}{\delta q^\beta} - L^\alpha + q^\beta_\gamma \frac{\partial L^\gamma}{\partial q^\beta_\alpha}. \tag{2.2}$$

Proof. Differentiating (2.2), we obtain

$$\frac{\partial \Phi^\alpha}{\partial x^\alpha} = q^\beta_\alpha \frac{\delta L^\alpha}{\delta q^\beta} + q^\beta \frac{\partial}{\partial x^\alpha}\left(\frac{\delta L^\alpha}{\delta q^\beta}\right) - \frac{\partial L^\alpha}{\partial q^\beta} q^\beta_\alpha - \frac{\partial L^\alpha}{\partial q^\beta_\gamma} q^\beta_{\gamma\alpha} + q^\beta_{\gamma\alpha} \frac{\partial L^\gamma}{\partial q^\beta_\alpha} + q^\beta_\gamma \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L^\gamma}{\partial q^\beta_\alpha}\right)$$

$$= q^\beta_\alpha \left(\frac{\partial L^\alpha}{\partial q^\beta} - \frac{\partial}{\partial x^\gamma}\frac{\partial L^\alpha}{\partial q^\beta_\gamma}\right) - q^\beta_\alpha \frac{\partial L^\alpha}{\partial q^\beta} - \frac{\partial L^\alpha}{\partial q^\beta_\gamma} q^\beta_{\gamma\alpha} + q^\beta_{\gamma\alpha} \frac{\partial L^\gamma}{\partial q^\beta_\alpha} + q^\beta_\gamma \frac{\partial}{\partial x^\alpha}\frac{\partial L^\gamma}{\partial q^\beta_\alpha} = -\frac{\partial L^\gamma}{\partial q^\beta_\alpha} q^\beta_{\alpha\gamma} + \frac{\partial L^\gamma}{\partial q^\beta_\alpha} q^\beta_{\gamma\alpha} = 0.$$

The theorem is proved.

In a similar way one can construct the conservation law (2.2), if L^α depends on second- and higher-order derivatives.

3. Examples. We further put $x^0 = t, x^1 = x$.

(1) *Generalized Korteweg–de Vries Equation* [8, 9]:

$$u_t + f(u)_x + u_{xxx} = 0. \tag{3.1}$$

Let $L_0 = u^2/2, L^1 = \overset{u}{\int} f(z)dz - u_x^2/2$. Then (3.1) is equivalent to the equation

$$\left(\frac{\delta L^0}{\delta u}\right)_t + \left(\frac{\delta L^1}{\delta u}\right)_x = 0.$$

(2) *Generalized Regularized Boussinesq Equation* [8–10]:

$$v_{tt} - v_{xx} - (f(v))_{xx} - v_{ttxx} = 0. \tag{3.2}$$

We write (3.2) as the system of equations

$$\left(\frac{\delta L^0}{\delta u}\right)_t + \left(\frac{\delta L^1}{\delta u}\right)_x = 0, \quad \left(\frac{\delta L^0}{\delta v}\right)_t + \left(\frac{\delta L^1}{\delta v}\right)_x = 0,$$

$$L^0 = uv, \ L^1 = \frac{v_t^2}{2} - \frac{v^2}{2} - \overset{v}{\int} f(z)dz - \frac{u^2}{2}.$$

(3) *Transverse Oscillations of Mass Points Suspended on a Weightless Unstrained Thread* (long-wave approximation) [11]. In dimensionless variables the equation of motion is as follows:

$$v_{tt} = \left(v^3 + \frac{\varepsilon^2}{4}(vv_x^2 + v^2 v_{xx})\right)_{xx} \tag{3.3}$$

(ε is a small parameter). Equation (3.3) is rewritten as the system

$$v_t - u_x = 0, \ u_t + p_x = 0, \qquad p = -\frac{\delta e}{\delta v}, \quad e = \frac{v^4}{4} - \frac{\varepsilon^2}{8} v^2 v_x^2. \tag{3.4}$$

Thus, one should set

$$L^0 = uv, \qquad L^1 = -u^2/2 - e(v, v_x),$$

and hence (3.4) is equivalent to the system

$$\left(\frac{\delta L^0}{\delta u}\right)_t + \left(\frac{\delta L^1}{\delta u}\right)_x = 0, \qquad \left(\frac{\delta L^0}{\delta v}\right)_t + \left(\frac{\delta L^1}{\delta v}\right)_x = 0.$$

(4) *Model of the Slug Flow Regime of a Gas–Liquid Mixture in the Neighborhood of a Thermodynamic Critical Point* (long-wave approximation) [12]. Equations of this model have the form (3.4).

(5) *Shallow Water Equations in the Boussinesq Approximation* [8]:

$$h_t + (hu)_z = 0, \qquad u_t + uu_z + gh_z + c_0^2 h_0 h_{zzz}/3 = 0. \tag{3.5}$$

Here t is the time, z is the Eulerian coordinate, h is the liquid depth, h_0 is the undisturbed depth, g is the acceleration of gravity; $c_0^2 = gh_0$. Let us introduce the mass Lagrangian coordinate x from the relation $\partial z/\partial x = h^{-1}$. Since

$$h_z = h_x \frac{\partial x}{\partial z} = hh_x, \qquad h_{zz} = h(hh_x)_x, \qquad h_{zzz} = h(h(hh_x)_x)_x,$$

the divergent form of the derivative is as follows:

$$\begin{aligned} h_z = (h^2/2)_x, \qquad h_{zzz} &= h(h(h^2/2)_{xx})_x = hh_x(h^2/2)_{xx} + h^2(h^2/2)_{xxx} \\ &= (h^2/2)_x(h^2/2)_{xx} + (h^2(h^2/2)_{xx})_x - (h^2)_x(h^2/2)_{xx} \\ &= (h^2(h^2/2)_{xx})_x - (1/4)(h^2)_{xx}(h^2)_x = \left\{h^2(h^2/2)_{xx} - (1/8)((h^2)_x)^2\right\}_x. \end{aligned}$$

After the passage to the Lagrangian variables Eqs. (3.5) become

$$\left(\frac{1}{h}\right)_t - u_x = 0, \quad u_t + \left\{\frac{gh^2}{2} + \frac{h_0c_0^2}{3}\left[h^2\left(\frac{h^2}{2}\right)_{xx} - \frac{1}{8}((h^2)_x)^2\right]\right\}_x = 0. \tag{3.5'}$$

Introducing the variable $v \equiv 1/h$, we have

$$(h^2)_x = -\frac{2v_x}{v^3}, \qquad (h^2)_{xx} = -2\left(\frac{v_x}{v^3}\right)_x = -\frac{2\left(v_{xx}v - 3v_x^2\right)}{v^4},$$

whence

$$h^2\left(\frac{h^2}{2}\right)_{xx} - \frac{1}{8}((h^2)_x)^2 = -\frac{v_{xx}v - 3v_x^2}{v^6} - \frac{v_x^2}{2v^6} = -\frac{v_{xx}}{v^5} + \frac{5v_x^2}{2v^6}.$$

Consequently, Eqs. (3.5′) become

$$v_t - u_x = 0, \quad u_t + \left\{\frac{g}{2v^2} + \frac{h_0c_0^2}{3}\left[\frac{5v_x^2}{2v^6} - \frac{v_{xx}}{v^5}\right]\right\}_x = 0. \tag{3.5''}$$

Introduce the function

$$e(v, v_x) = \frac{g}{2v} - \frac{h_0c_0^2}{6}\frac{v_x^2}{v^5}.$$

Then

$$\begin{aligned} -\frac{\delta e}{\delta v} &= -\left[-\frac{g}{2v^2} + \frac{5h_0c_0^2}{6}\frac{v_x^2}{v^6} + \frac{\partial}{\partial x}\left(\frac{h_0c_0^2}{3}\frac{v_x}{v^5}\right)\right] \\ &= -\left[-\frac{g}{2v^2} + \frac{5h_0c_0^2}{6}\frac{v_x^2}{v^6} + \frac{h_0c_0^2}{3}\frac{v_{xx}}{v^5} - \frac{5h_0c_0^2}{3}\frac{v_x^2}{v^6}\right] = \frac{g}{2v^2} + \frac{5h_0c_0^2}{6}\frac{v_x^2}{v^6} - \frac{h_0c_0^2}{3}\frac{v_{xx}}{v^5}. \end{aligned}$$

Thus, Eqs. (3.5″) are written as follows [12]:

$$v_t - u_x = 0, \quad u_t + p_x = 0, \quad p = -\frac{\delta e}{\delta v}, \quad e = \frac{g}{2v} - \frac{h_0c_0^2}{6}\frac{v_x^2}{v^5}. \tag{3.5'''}$$

(6) *Granular Media (One-Dimensional Chains)*. In the case of propagation of nonlinear waves in discrete one-dimensional media with the interaction of particles according to the generalized Hertz law, the following

equation [13] in the long-wave approximation arises:

$$w_{tt} = -c_n^2 \left\{ (-w_x)^n + \frac{na^2}{6(n+1)} \left[(-w_x)^{\frac{n-1}{2}} ((-w_x)^{\frac{n+1}{2}})_{xx} \right] \right\}_x, \tag{3.6}$$

where a is the distance between the particles; w is the displacement from equilibrium; c_n is a parameter having dimension of velocity; $n > 1$ is the exponent of nonlinearity in the generalized Hertz law ($F \sim \delta^n$, where F is the force of interaction between the particles and δ is the approach of the particles). Reasoning from physical considerations, we consider only solutions with $w_x < 0$.

Let us change the variables $w_t \equiv u,\ w_x \equiv v$. Then (3.6) is equivalent to the system

$$v_t - u_x = 0, \qquad u_t + p_x = 0, \qquad p = c_n^2 \left\{ (-v)^n + \frac{na^2}{6(n+1)} (-v)^{\frac{n-1}{2}} \left((-v)^{\frac{n+1}{2}} \right)_{xx} \right\}.$$

Then we have

$$p = c_n^2 \left\{ (-v)^n + \frac{na^2}{12} (-v)^{\frac{n-1}{2}} \left((-v)^{\frac{n-1}{2}} (-v_x) \right)_x \right\},$$

$$p = c_n^2 \left\{ (-v)^n + \frac{n(n-1)a^2}{24} (-v)^{n-2} v_x^2 - \frac{na^2}{12} (-v)^{n-1} v_{xx} \right\}.$$

We now take

$$e(v, v_x) = c_n^2 \left\{ \frac{(-v)^{n+1}}{n+1} + A(-v)^{n-1} v_x^2 \right\}.$$

Define the constant A in such a way that

$$p = -\left\{ \frac{\partial e}{\partial v} - \frac{\partial}{\partial x} \left(\frac{\partial e}{\partial v_x} \right) \right\},$$

$$p = -c_n^2 \left\{ -(-v)^n - A(n-1)(-v)^{n-2} v_x^2 - \frac{\partial}{\partial x} (2A(-v)^{n-1} v_x) \right\}$$

$$= c_n^2 \left\{ (-v)^n - A(n-1)(-v^{n-2}) v_x^2 + 2A(-v)^{n-1} v_{xx} \right\}.$$

Comparing this expression with the one obtained earlier, we find that $A = -na^2/24$.

Thus, system (3.6) is also written as

$$v_t - u_x = 0, \quad u_t + p_x = 0, \quad p = -\frac{\delta e}{\delta v}, \quad e = c_n^2 \left\{ \frac{(-v)^{n+1}}{n+1} - \frac{na^2}{24} (-v)^{n-1} v_x^2 \right\}. \tag{3.6$'$}$$

It should be noted that when $n = 3$ system (3.6′) coincides with (3.4).

(7) *Van der Waals–Korteweg Theory of Capillarity* [14–16]. Investigation of liquid–gas isothermic phase transitions yields the system

$$v_t - u_x = 0, \quad u_t + p_x = 0, \quad p = -\frac{\delta e}{\delta v}, \quad e = -\int^v p_0(w)dw + \frac{cv_x^2}{2}, \quad p_0(v) = \frac{RT}{v-b} - \frac{a}{v^2}, \tag{3.7}$$

where x is the mass Lagrangian coordinate, v is the specific volume of the mixture, u is the velocity of the mixture, R is the gas constant, T is the temperature, and a, b, and c are positive constants.

We note that systems (3.4), (3.5‴), (3.6′), and (3.7) are distinguished only by different functions $e(v,\ v_x)$ and are reminiscent of the equations of gasdynamics in the mass Lagrangian coordinates. The difference is that the pressure p here depends not only on the specific volume v, but also on the derivatives v_x and v_{xx}. For the systems of equations of this type the most natural is the problem on the structure of traveling waves rather than the problem of the decay of an arbitrary discontinuity. In [12] the problem of the stability of periodic traveling waves for an arbitrary function $e(v,\ v_x)$ is studied. (Periodic waves are stable or unstable depending on whether the appropriate system of modulation equations is of hyperbolic or mixed type. The stability of periodic waves in a particular case is also studied in [17].)

(8) *Bubbly Liquid.* The equations of a bubbly liquid were derived independently by different authors [18–20]. The one-dimensional equations of motion disregarding the dissipation, mass exchange, fragmentation of bubbles, surface tension, and compressibility of the carrier phase are of the form

$$\rho_t + (\rho u)_z = 0, \quad u_t + uu_z + \frac{1}{\rho} p_z = 0, \quad R\frac{d^2R}{dt^2} + \frac{3}{2}\left(\frac{dR}{dt}\right)^2 = \frac{1}{\rho_e}(p_g(R) - p), \quad N_t + (uN)_z = 0. \tag{3.8}$$

Here z is the Eulerian coordinate; $d/dt = \partial/\partial t + u\partial/\partial z$; $\rho = \alpha_l\rho_l + \alpha_g\rho_g$ is the density of the mixture (the subscripts l and g stand for liquid and gas respectively); α_l and α_g ($\alpha_l + \alpha_g = 1$) are the volume concentrations of the liquid and the gas, respectively; p is the pressure in the mixture, R is the averaged radius of bubbles; $p_g(R)$ is the pressure inside a gas bubble; N is the number of bubbles per unit volume of the mixture. The volume concentration α_g and the density ρ_g are defined by the formulas

$$\alpha_g = \frac{4}{3}\pi R^3 N, \qquad \rho_g = \frac{3m_g}{4\pi R^3},$$

where m_g = const is the mass of the gas inside a bubble.

It was shown [21] that if the mass Lagrangian coordinate x is introduced, Eqs. (3.8) become

$$v_t - u_x = 0, \qquad u_t + p_x = 0, \qquad p = -\frac{\delta e(v, v_t)}{\delta v} = -\left[\frac{\partial e}{\partial v} - \frac{\partial}{\partial t}\frac{\partial e}{\partial v_t}\right], \tag{3.8$'$}$$

$$e(v, v_t) = e_g(v) - 2\pi n \rho_l R^3 R_t^2, \quad v \equiv \frac{1}{\rho_l} + \frac{4}{3}\pi R^3 n, \quad n \equiv \frac{N}{\rho} = \text{const}, \quad de_g(v) + p_g(R)dv = 0.$$

In [21] the modulation equations for system (3.8′) are constructed for an arbitrary function $e(v, v_t)$. A particular case of the system of modulation equations is obtained in [22] where the oscillations of bubbles near the resonance are considered. In [23] the effects of dissipation are taken into account when deriving the modulation equations.

4. Media Whose Equations of State Depend on $\nabla\rho$. Let us consider the governing equations of a medium whose specific internal energy ε is a function of density, density gradient, and entropy.

Mechanical System. Let us first consider the case of a mechanical system where ε depends only on ρ and on $\nabla\rho$ [a multi-dimensional analog of system (3.4)]. Since ε is a Galilean invariant, we have

$$\varepsilon = \varepsilon(\rho, |\nabla\rho|^2), \ |\nabla\rho|^2 \equiv \sum_i |\partial\rho/\partial x^i|^2.$$

The equations of such a medium are constructed in [15, 16, 24] (see also references therein). In this paper the derivation of equations is given to provide a complete presentation, and also to distinguish the role of variational derivatives in determining the effective pressure P [see below (4.6)] and to emphasize some other important aspects.

In what follows, the Latin indices take on the values $1, \ldots, n$, and $x = (x^i) \in R^n$. For the derivatives we also used the notation $\varphi_t \equiv \partial\varphi/\partial t$, $\varphi_k \equiv \partial\varphi/\partial x^k$.

Let the Lagrangian coordinates of the particles be $\xi(t, x) = (\xi^a)$, $\xi^a_k = \partial\xi^a/\partial x^k$. The matrix that is the inverse of the matrix (ξ^a_k) is denoted by (x^k_a) so that

$$x^k_a \xi^b_k = \delta^b_a, \quad x^k_a \xi^a_i = \delta^k_i, \tag{4.1}$$

where δ^b_a is the Kronecker symbol. After differentiation with respect to ξ^c_m and multiplication by x^i_b, it follows from (4.1) that

$$\frac{\partial x^i_a}{\partial \xi^c_m} = -x^m_a x^i_c. \tag{4.2}$$

Since (ξ^a) are the Lagrangian coordinates,

$$\frac{\partial\xi^a}{\partial t} + v^k\frac{\partial\xi^a}{\partial x^k} = 0,$$

which, in view of (4.2), yields the equalities

$$v^i = -x^i_a \xi^a_t; \tag{4.3}$$

$$\frac{\partial v^k}{\partial \xi^a_i} = -x^k_a v^i. \tag{4.4}$$

Denote

$$p \equiv \rho^2 \frac{\partial \varepsilon}{\partial \rho} = \rho \frac{\partial(\rho\varepsilon)}{\partial \rho} - \rho\varepsilon, \qquad \frac{1}{2}\lambda \equiv \frac{\partial \varepsilon}{\partial |\nabla \rho|^2}. \tag{4.5}$$

To derive the equations, we apply a variational principle (see also [25]). We take the functional

$$J' = \int\limits_{t_0}^{t_1} \int\limits_{R^n} \left\{ \rho\left[\frac{v^2}{2} - \varepsilon\right] + \varphi\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^k}(\rho v^k)\right]\right\} dxdt.$$

Here $\varphi(t, x)$ is the Lagrangian multiplier corresponding to the introduction of the law of conservation of mass as a constraint. Since the divergent summand does not influence the form of the Euler equations, J' can be replaced by a functional that is more convenient for calculations:

$$J = \int\limits_{t_0}^{t_1} \int\limits_{R^n} L dxdt, \qquad L \equiv \rho\left\{\frac{v^2}{2} - \varepsilon - \frac{d\varphi}{dt}\right\}, \qquad \frac{d\varphi}{dt} \equiv \frac{\partial \varphi}{\partial t} + v^k \frac{\partial \varphi}{\partial x^k}.$$

Computation of Variational Derivatives. We define

$$E_\varphi \equiv \frac{\delta L}{\delta \varphi} = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^k}(\rho v^k), \quad E_\rho \equiv \frac{\delta L}{\delta \rho} = \frac{1}{\rho}(L - P), \quad P \equiv p - \rho \frac{\partial}{\partial x^i}\left(\rho\lambda \frac{\partial \rho}{\partial x^i}\right) = \rho \frac{\delta(\rho\varepsilon)}{\delta \rho} - \rho\varepsilon. \tag{4.6}$$

Denote $E_a \equiv \delta L/\delta \xi^a$ and calculate $E_j \equiv \xi^a_j E_a$:

$$E_j = -\xi^a_j \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \xi^a_t}\right) - \xi^a_j \frac{\partial}{\partial x^k}\left(\frac{\partial L}{\partial \xi^a_k}\right) = -\frac{\partial}{\partial t}\left(\xi^a_j \frac{\partial L}{\partial \xi^a_t}\right) - \frac{\partial}{\partial x^k}\left(\xi^a_j \frac{\partial L}{\partial \xi^a_k} - L\delta^{jk}\right) - R_j,$$

$$R_j = \frac{\partial L}{\partial \rho}\frac{\partial \rho}{\partial x^j} + \frac{\partial L}{\partial \rho_i}\frac{\partial \rho_i}{\partial x_j} + \frac{\partial L}{\partial \varphi_t}\frac{\partial \varphi_t}{\partial x^j} + \frac{\partial L}{\partial \varphi_k}\frac{\partial \varphi_k}{\partial x^j} = E_\rho \frac{\partial \rho}{\partial x^j} + \frac{\partial}{\partial x^i}\left(\frac{\partial \rho}{\partial x^j}\frac{\partial L}{\partial \rho_i}\right) - \left\{\frac{\partial}{\partial t}(\rho\varphi_j) + \frac{\partial}{\partial x^k}(\rho v^k \varphi_j)\right\} + \varphi_j E_\varphi.$$

Taking into account (4.3) and (4.4), we find

$$\xi^a_j \frac{\partial L}{\partial \xi^a_t} = -\rho v^j + \rho\varphi_j, \qquad \xi^a_j \frac{\partial L}{\partial \xi^a_k} = -\rho v^j v^k + \rho v^k \varphi_j.$$

Invariance and the Law of Conservation of Momentum. The functional J is invariant with respect to spatial translations, i.e., the groups of transformations G_j with the operator $\partial/\partial x^j$ [26]. According to the Noether theorem this leads to the divergence of the expression

$$\mathbf{J}^j \equiv \varphi_j E_\varphi + \rho_j E_\rho + \xi^a_j E_a \equiv \varphi_j E_\varphi + \rho_j E_\rho + E_j.$$

Gathering the relations obtained earlier we find

$$\mathbf{J}^j = \frac{\partial}{\partial t}(\rho v^j) + \frac{\partial}{\partial x^k}\left\{\rho v^j v^k + L\delta^{jk} + \rho\lambda \frac{\partial \rho}{\partial x^k}\frac{\partial \rho}{\partial x^j}\right\}.$$

Invariance and the Law of Energy Conservation. The functional J is invariant with respect to time shifts, i.e., the group G_0 with the operator $\partial/\partial t$. Following the Noether theorem this results in the divergence of the expression

$$\mathbf{E} \equiv -\varphi_t E_\varphi - \rho_t E_\rho - \xi^a_t E_a.$$

After calculations we obtain

$$\mathbf{E} = \frac{\partial}{\partial t}\left\{\rho\left(\varepsilon + \frac{v^2}{2}\right)\right\} + \frac{\partial}{\partial x^k}\left\{\rho v^k\left(\varepsilon + \frac{v^2}{2}\right) + v^k L - \rho\lambda \frac{\partial \rho}{\partial t}\frac{\partial \rho}{\partial x^k}\right\}.$$

Final System. The Euler equations have the form $E_\varphi = 0$, $E_\rho = 0$, and $E_a = 0$. Hence, it follows that $\mathbf{J}^j = 0$ and $\mathbf{E} = 0$. The equation $E_\rho = 0$ gives the relation

$$L = P. \tag{4.7}$$

Taking into account Eq. (4.7), we arrive at the system of equations

$$\mathrm{M} \equiv \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^k}(\rho v^k) = 0; \tag{4.8}$$

$$\mathbf{J}^j \equiv \frac{\partial}{\partial t}(\rho v^j) + \frac{\partial}{\partial x^k}(\rho v^j v^k + \Pi^{jk}) = 0; \tag{4.9}$$

$$\Pi^{jk} \equiv P\delta^{jk} + \rho\lambda \frac{\partial \rho}{\partial x^j}\frac{\partial \rho}{\partial x^k}; \tag{4.10}$$

$$\mathbf{E} \equiv \frac{\partial}{\partial t}\left\{\rho\left(\varepsilon + \frac{v^2}{2}\right)\right\} + \frac{\partial}{\partial x^k}\left\{\rho v^k\left(\varepsilon + \frac{v^2}{2}\right) + v^j\Pi^{jk} - \rho\lambda\frac{\partial \rho}{\partial x^k}\frac{d\rho}{dt}\right\} = 0. \tag{4.11}$$

Equation (4.11) is the consequence of (4.6) and (4.8)–(4.10). The important equality

$$\mathbf{E} = (h - v^2/2)\mathrm{M} + v^j\mathbf{J}^j \tag{4.12}$$

is valid, where

$$h \equiv \delta(\rho\varepsilon)/\delta\rho. \tag{4.13}$$

Relations (4.12) and (4.13) have a characteristic property: they coincide in form with the appropriate relations for a barotropic liquid when the ordinary derivative $\partial(\rho\varepsilon)/\partial\rho$ is replaced by a variational one [cf. also (4.5) and (4.6)]

Taking into Account Heat. Let a local state of the physical system in question be described by the set of variables ρ, $\nabla\rho$, s, $v = (v^k)$ (s is the specific entropy) and the equation of state $\varepsilon = \varepsilon(\rho, |\nabla\rho|^2, s)$ be given. We assume that

$$p \equiv \rho^2\frac{\partial\varepsilon}{\partial\rho} = \rho\frac{\partial(\rho\varepsilon)}{\partial\rho} - \rho\varepsilon, \quad \frac{\lambda}{2} \equiv \frac{\partial\varepsilon}{\partial|\nabla\rho|^2}, \quad T \equiv \frac{\partial\varepsilon}{\partial s}, \quad P \equiv \rho^2\frac{\partial\varepsilon}{\partial\rho} - \rho\frac{\partial}{\partial x^j}\left(\rho\lambda\frac{\partial\rho}{\partial x^i}\right) = \rho\frac{\delta(\rho\varepsilon)}{\delta\rho} - \rho\varepsilon.$$

Let us introduce the Galilean vectors

$$V = (V^\alpha) \equiv (1, v) = (1, v^1, \dots, v^n), \quad W = (W^\alpha) \equiv (0, \rho_1, \dots, \rho_n),$$

where $\rho_j \equiv \partial\rho/\partial x^j$. Then we need a general form of the divergent Galilean scalar, i.e., the expression

$$X \equiv \frac{\partial}{\partial x^\alpha}Y^\alpha, \quad x^0 \equiv t,$$

where Y^α are assumed to depend only on ρ, s, V, and W and X is a Galilean invariant or scalar. Following [27]

$$Y = AV + BW + \text{const}. \tag{4.14}$$

Here A and B depend only on ρ, $|\nabla\rho|^2$, and s.

Now the problem is reduced to the search for a stationary (extremum) point of the functional

$$J' \equiv \int_{t_0}^{t_1}\int_{R^n}\rho\left(\frac{v^2}{2} - \varepsilon\right)dx\,dt$$

with imposed constraints. As the constraints we take the law of mass conservation (4.8) and the law of entropy conservation. The latter should be, first, an exact conservation law (cf. definition of the system with dispersion in Section 1), i.e., should have the form $\partial Y^\alpha/\partial x^\alpha = 0$, and, second, the left-hand side of the equality should be a Galilean scalar. Since we consider the systems whose local state is characterized by the set (ρ, $\nabla\rho$, s, v), Y^α is assumed to depend only on these variables. Hence, (4.14) holds. Finally, according to its physical sense, the law of entropy conservation should be written as $\partial(\rho s)/\partial t + \partial(\ldots)^k/\partial x^k = 0$, whence $A = \rho s$. We finally take

$$\frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x^k}\left(\rho s v^k + \rho\zeta\frac{\partial\rho}{\partial x^k}\right) = 0, \tag{4.15}$$

where $\zeta = \zeta(\rho, |\nabla\rho|^2, s)$. Assume first that $\zeta = 0$. In this case, we obtain equations having the form of Eqs. (4.8)–(4.11), where $\varepsilon = \varepsilon(\rho, |\nabla\rho|^2, s)$, which are supplemented by the equation

$$\mathrm{S} \equiv \frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x^k}(\rho s v^k) = 0. \tag{4.16}$$

One of Eqs. (4.8)–(4.11), (4.16) is the consequence of the others. This follows from the equality similar to (4.12):

$$\mathrm{E} = \left(\gamma - \frac{v^2}{2}\right)\mathrm{M} + v^j \mathrm{J}^j + T\mathrm{S}; \tag{4.17}$$

$$\gamma \equiv \left.\frac{\delta(\rho\varepsilon)}{\delta\rho}\right|_{(\rho s)} = \varepsilon + P/\rho - Ts. \tag{4.18}$$

If $\varepsilon = \varepsilon(\rho, s) = \varepsilon(\rho, (\rho s)/\rho)$, then

$$\gamma = \left.\frac{\partial(\rho\varepsilon)}{\partial\rho}\right|_{(\rho s)} = \varepsilon + p/\rho - Ts, \tag{4.19}$$

i.e., γ is the specific Gibbs potential. Thus, (4.18) is a natural generalization of the specific Gibbs potential (4.19) for the class of systems under consideration. The equality

$$P = \rho\left.\frac{\delta(\rho\varepsilon)}{\delta\rho}\right|_{(\rho s)} + (\rho s)\left.\frac{\delta(\rho\varepsilon)}{\delta(\rho s)}\right|_{\rho} - \rho\varepsilon \tag{4.20}$$

is valid, which is apparently nothing but a variational analog of the Legendre transform. The symbol of the type $\delta(\rho\varepsilon)/\left.\delta(\rho s)\right|_\rho$ means that when the variational derivative is calculated the variable ρ remains invariant. In this case,

$$\left.\frac{\delta(\rho\varepsilon)}{\delta(\rho s)}\right|_\rho = \frac{\partial\varepsilon}{\partial s} = T, \quad \left.\frac{\delta(\rho\varepsilon)}{\delta\rho}\right|_{(\rho s)} = \gamma = \varepsilon + \frac{P}{\rho} - Ts.$$

It seems likely that transformation (4.20) is reversible to a certain reasonable degree, although this question still remains to be solved. The fundamental equality (4.17) can also be written as

$$\mathrm{S} = -\frac{1}{T}\left(\gamma - \frac{v^2}{2}\right)\mathrm{M} - \frac{1}{T}v^j\mathrm{J}^j + \frac{1}{T}\mathrm{E}, \tag{4.21}$$

whence it is evident that $1/T$ is the integrating multiplier in the energy conservation law as in the classical mechanics of continua.

Equation (4.21) is properly one of the most exact and general expressions of the second principle of thermodynamics (excluding what is involved in the notions of irreversibility). Let us take the classical differential Gibbs form

$$d\varepsilon = T\,ds - p\,d(1/\rho), \tag{4.22}$$

where $\varepsilon = \varepsilon(\rho, s)$.

It is equivalent to the differential form

$$d(\rho\varepsilon) = \gamma\,d\rho + T\,d(\rho s), \tag{4.23}$$

$$\gamma = \partial(\rho\varepsilon)/\left.\partial\rho\right|_{(\rho s)} = \varepsilon + p/\rho - Ts, \quad T = \partial(\rho\varepsilon)/\left.\partial(\rho s)\right|_\rho.$$

In the context of the equations of classical gas dynamics Eqs. (4.22), (4.23), and (4.17) are equivalent. Therefore, each of them is an original representation of the second principle of thermodynamics. But (4.17) is in some respects a more general expression than (4.23) since it makes sense and for nonclassical media can also be used in determining analogs of the Gibbs potential γ and the temperature T. To distinguish more clearly the far reaching structural relationship of the classical and new variational formulas, we introduce the variational thermodynamic form

$$\delta(\rho\varepsilon) = \gamma\delta\rho + T\delta(\rho s), \tag{4.24}$$

which is given the following natural meaning: by definition it is equivalent to the relations

$$\gamma = \left.\frac{\delta(\rho\varepsilon)}{\delta\rho}\right|_{(\rho s)}, \qquad T = \left.\frac{\delta(\rho\varepsilon)}{\delta(\rho s)}\right|_{\rho}.$$

The structure of expressions (4.23) and (4.24) is identical.

Thus, the fundamental relations of classical continuum mechanics and thermodynamics are extended to the class of media with dispersion with the ordinary derivative replaced by the variational derivative.

General Note. Generally speaking, the equation of state of the type $\varepsilon = \varepsilon(\rho, |\nabla\rho|^2, s)$ can be obtained in two ways.

First, it can be obtained by means of appropriate averaging of the initial system with high-frequency oscillations (fluctuations). In this case $T = \partial\varepsilon/\partial s$ may not have a direct meaning of physical temperature. But it is its complete analog, so that $1/T$ is the integrating multiplier of E in (4.21). This analogy is useful in studying media with nonconvex equations of state where phase transitions or other peculiarities can occur, in constructing dissipative systems, and in extending the Onsager relations to systems with dispersion (see [27, 28]).

Second, one can reason from generalization of a simple classical situation in which the local state of the physical system in question was originally characterized by the set of variables $\rho, s, v = (v^k)$ and was given by the equation of state $\varepsilon = \varepsilon_0(\rho, s)$, with $T = \partial\varepsilon_0/\partial s$ having the meaning of physical temperature. If this description turned out to be inadequate in one or another relation, one can try to extend the initial set of variables determining the state of the system at the point (t, x) by introducing additionally some variables, for example, $\nabla\rho$, and simultaneously generalizing the equation of state, i.e., specifying it as $\varepsilon = \varepsilon(\rho, \nabla\rho, s)$ (in effect this is precisely often the case). In this case the unique correct definition for temperature is $T = \partial\varepsilon/\partial s$, which restricts the choice of possible equations of state.

If the derivatives $\partial s/\partial x$ are introduced into the local set of variables governing the state of the system, one should pass to the variational derivative when calculating temperature in terms of $(\rho\varepsilon)$. In all cases the quantity that is the reverse of the temperature should be an integrating multiplier of E in the expression of the type (4.21).

General Case. Let $\zeta \neq 0$. We take the functional

$$J' \equiv \int_{t_0}^{t_1}\int_{R^n} \left\{\rho\left[\frac{v^2}{2} - \varepsilon\right] + \varphi\left[\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x^k}(\rho v^k)\right] + \psi\left[\frac{\partial(\rho s)}{\partial t} + \frac{\partial}{\partial x^k}\left(\rho s v^k + \rho\zeta\frac{\partial\rho}{\partial x^k}\right)\right]\right\} dx\,dt,$$

allowing for the constraint (4.15). After calculations we obtain the system of equations

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x^k}\left(\rho v^k\right) = 0; \tag{4.25}$$

$$\frac{\partial}{\partial t}\left(\rho v^j\right) + \frac{\partial}{\partial x^k}\left\{\rho v^j v^k + (P + G)\delta^{jk} + \rho\lambda\frac{\partial\rho}{\partial x^i}\frac{\partial\rho}{\partial x^j} + G^{jk}\right\} = 0,$$

$$G \equiv \rho\left\{\frac{\partial\zeta}{\partial\rho}\rho_k\psi_k - \frac{\partial}{\partial x^i}\left(\rho\zeta\psi_i + 2\rho\frac{\partial\zeta}{\partial|\nabla\rho|^2}\rho_i\rho_k\psi_k\right)\right\}, \tag{4.26}$$

$$G^{jk} \equiv 2\zeta\left(\rho_j\psi_k + \rho_k\psi_j\right) + 2\rho\frac{\partial\zeta}{\partial|\nabla\rho|^2}\rho_j\rho_k\rho_i\psi_i;$$

$$\frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x^k}\Big(\rho s v^k + \rho\zeta\frac{\partial\rho}{\partial x^k}\Big) = 0; \tag{4.27}$$

$$\frac{d\psi}{dt} + \frac{\partial\zeta}{\partial s}\frac{\partial\rho}{\partial x^k}\frac{\partial\psi}{\partial x^k} = -T. \tag{4.28}$$

The law of energy conservation for system (4.25)–(4.28) is as follows:

$$\begin{aligned} &\frac{\partial}{\partial t}\Big\{\rho\Big(\varepsilon + \frac{v^2}{2} + \zeta\rho_i\psi_i\Big)\Big\} + \frac{\partial}{\partial x^k}\Big\{\rho v^k\Big(\varepsilon + \frac{v^2}{2} + \zeta\rho_i\psi_i\Big) \\ &+ v^k(P + G) - \frac{\partial\rho}{\partial t}\Big(\rho\lambda\rho_k\Big) + \rho\zeta\Big(\rho_k\psi_t - \rho_t\psi_k\Big) - 2\frac{\partial\zeta}{\partial|\nabla\rho|^2}\rho_k\rho_i\psi_i\Big\} = 0. \end{aligned} \tag{4.29}$$

Equations (4.25)–(4.29) are complicated. But in some cases they are considerably simplified. For example, if ζ depends only on ρ, we have

$$G = -\rho\zeta\frac{\partial}{\partial x^i}\Big(\rho\frac{\partial\psi}{\partial x^i}\Big), \quad \frac{d\psi}{dt} = -T.$$

The case $\zeta \neq 0$ is considered to demonstrate that in systems with dispersion more general expressions similar to (4.15) are possible along with the traditional conservation laws of the form (4.16). In some cases they can probably have physical meaning and describe new physical phenomena.

5. Media with Equations of State Depending on $\dot\rho$. Let us now consider a medium for which the set of variables governing the state of the system is as follows: $\rho, \dot\rho \equiv d\rho/dt$, $s, v = (v^k)$. We take the functional

$$J \equiv \int_{t_0}^{t_1}\int_{R^n} \Big\{\rho\Big[\frac{v^2}{2} - e\Big] + \varphi\Big[\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x^k}(\rho v^k)\Big] + \psi\Big[\frac{\partial(\rho s)}{\partial t} + \frac{\partial}{\partial x^k}(\rho s v^k)\Big]\Big\}\,dx dt,$$

where $e = e(\rho, \dot\rho, s)$. As a constraint we take here the energy and entropy conservation laws.

After calculations we obtain the system

$$\mathbf{M} \equiv \frac{\partial\rho}{\partial t} + \frac{\partial(\rho v^k)}{\partial x^k} = 0; \tag{5.1}$$

$$\mathbf{J}^j \equiv \frac{\partial}{\partial t}(\rho v^j) + \frac{\partial}{\partial x^k}(\rho v^j v^k + P\delta^{jk}) = 0; \tag{5.2}$$

$$P \equiv p - \rho\Big\{\frac{\partial}{\partial t}\Big[\frac{\partial(\rho e)}{\partial\dot\rho}\Big] + \frac{\partial}{\partial x^k}\Big[\frac{\partial(\rho e)}{\partial\dot\rho}v^k\Big]\Big\}; \tag{5.3}$$

$$p \equiv \rho^2\frac{\partial e}{\partial\rho} = \rho\frac{\partial(\rho e)}{\partial\rho} - \rho e; \tag{5.4}$$

$$\mathbf{S} \equiv \frac{\partial}{\partial t}(\rho s) + \frac{\partial}{\partial x^k}(\rho s v^k) = 0. \tag{5.5}$$

The law of energy conservation follows from the above equations

$$\mathbf{E} \equiv \frac{\partial}{\partial t}\Big\{\rho\Big(\varepsilon + \frac{v^2}{2}\Big)\Big\} + \frac{\partial}{\partial x^k}\Big\{\rho v^k\Big(\varepsilon + \frac{v^2}{2}\Big) + v^k P\Big\} = 0; \tag{5.6}$$

$$\varepsilon \equiv e - \dot\rho\frac{\partial e}{\partial\dot\rho}. \tag{5.7}$$

Expression (5.7) is a modification of the Legendre transform. We pass from the set of variables $(\rho, \dot\rho, s;\ e(\rho, \dot\rho, s))$ to the set $(\rho, \sigma, s;\ \varepsilon(\rho, \sigma, s))$, where $\sigma \equiv -\partial e/\partial\dot\rho$ assuming that the last equation is uniquely solvable so that the transformation $(\rho, \dot\rho, s) \to (\rho, \sigma, s)$ is one-to-one. The relations characteristic of the Legendre transforms

$$\frac{\partial\varepsilon}{\partial\rho}(\rho, \sigma, s) = \frac{\partial e}{\partial\rho}(\rho, \dot\rho, s), \qquad \frac{\partial\varepsilon}{\partial s}(\rho, \sigma, s) = \frac{\partial e}{\partial s}(\rho, \dot\rho, s), \qquad \sigma = -\frac{\partial e}{\partial\dot\rho}(\rho, \dot\rho, s),$$

$$\varepsilon = e - \dot\rho\frac{\partial e}{\partial\dot\rho} = e + \dot\rho\sigma, \qquad \dot\rho = \frac{\partial\varepsilon}{\partial\sigma}(\rho,\sigma,s), \qquad e = \varepsilon - \sigma\frac{\partial\varepsilon}{\partial\sigma} = \varepsilon - \sigma\dot\rho.$$

are valid. In particular, if $e = \varepsilon_0(\rho,s) - \mu(\rho,s)(\dot\rho)^2/2$, then $\sigma = \mu\dot\rho$ and $\varepsilon = \varepsilon_0(\rho,s) + \sigma^2/2\mu$.

Expression (5.3) can be written as

$$P = \rho\frac{\delta(\rho e)}{\delta\rho} - \rho e, \qquad e = e(\rho,\dot\rho,s). \tag{5.8}$$

Thus, system (5.1)–(5.6) has the structure of the classical equations of gasdynamics with the ordinary derivative $\partial/\partial\rho$ replaced by a variational one in defining the pressure P. Relation (5.8) is a variational analog of the Legendre transform. The following relations similar to (4.17) and (4.18) are also true:

$$\mathbf{E} = \Big(\gamma - \frac{v^2}{2}\Big)\mathbf{M} + v^j\mathbf{J}^j + T\mathbf{S}; \tag{5.9}$$

$$\gamma \equiv \frac{\delta(\rho e)}{\delta\rho}\bigg|_{(\rho s)} = e + \frac{P}{\rho} - Ts, \qquad T \equiv \frac{\partial e}{\partial s} = \frac{\partial\varepsilon}{\partial s} = \frac{\delta(\rho e)}{\delta(\rho s)}\bigg|_{\rho}. \tag{5.10}$$

In calculating γ in (5.10) (ρe) is considered as a function of the variables $\rho, \dot\rho$, and ρs, i.e., $e = e(\rho,\dot\rho,(\rho s)/\rho)$.

The equality analogous to (4.20)

$$P = \rho\,\frac{\delta(\rho e)}{\delta\rho}\bigg|_{(\rho s)} + (\rho s)\,\frac{\delta(\rho e)}{\delta(\rho s)}\bigg|_{\rho} - \rho e \tag{5.11}$$

holds true.

The coincidence of the fundamental relations (4.17), (4.18) and (5.9), (5.10), (4.20), and (5.11) cannot be occasional. This reveals the general fundamental structure of correctly constructed equations for media with dispersion, which allows one to extend the most important notions of the classical thermodynamics (temperature, Gibbs's potential, etc.) to media with equations of state depending on derivatives, to generalize to them the theory of phase transitions for nonconvex equations of state, and to construct correctly dissipative structures (see also [27–29]).

Mechanical System. If e depends only on ρ and $\dot\rho$, this yields the system of equations (5.1)–(5.3), (5.6). The equations for the second-order approximation of the long-wave theory were constructed in [30] without using the hypothesis on the smallness of the wave amplitude. In the case of a horizontal bottom and ignoring the bottom friction for a flow in the plan $[x = (x^1, x^2)]$ they have the form

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x^k}(hV^k) = 0; \tag{5.12}$$

$$\frac{\partial}{\partial t}(hV^j) + \frac{\partial}{\partial x^k}(\rho V^j V^k + P\delta^{jk}) = 0, \qquad P \equiv g\frac{h^2}{2} + \pi; \tag{5.13}$$

$$\frac{\partial}{\partial t}(hW) + \frac{\partial}{\partial x^k}(hWV^k) = \frac{3}{2h}\pi; \tag{5.14}$$

$$\frac{\partial V^k}{\partial x^k} + \frac{2}{h}W = 0. \tag{5.15}$$

Here $h(t,x)$ is the flow depth; $V(t,x) = (V^1, V^2)$ are the components of the mean velocity in the x^1 and x^2-directions, respectively; W is the mean vertical velocity:

$$W = \frac{1}{h}\int_0^h w(t,x,z)dz;$$

the value π characterizes the deviation of pressure from the hydrostatic law. The energy conservation law

$$\frac{\partial}{\partial t}\left\{h\left(\varepsilon+\frac{v^2}{2}\right)\right\}+\frac{\partial}{\partial x^k}\left\{hV^k\left(\varepsilon+\frac{v^2}{2}\right)+V^kP\right\}=0, \quad \varepsilon\equiv\frac{1}{2}gh+\frac{2}{3}W^2 \tag{5.16}$$

follows from (5.12)–(5.15).

System (5.12)–(5.15) is a set of five equations for five unknown quantities: h, V^1, V^2, W, and π. We transform the system. From (5.12) and (5.15) we find

$$W=\frac{\dot{h}}{2}, \qquad \varepsilon=\frac{1}{2}gh+\frac{1}{3}\frac{(\dot{h})^2}{2}, \qquad \sigma=\frac{1}{3}\dot{h}.$$

Finally we obtain

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x^k}\left(hV^k\right)=0; \tag{5.17}$$

$$\frac{\partial}{\partial t}(hV^j)+\frac{\partial}{\partial x^k}(hV^jV^k+P\delta^{jk})=0; \tag{5.18}$$

$$P\equiv\frac{1}{2}gh^2+h\left\{\frac{\partial}{\partial t}(h\sigma)+\frac{\partial}{\partial x^k}\left(h\sigma V^k\right)\right\}. \tag{5.19}$$

The unknown quantities in (5.17) and (5.18) are h and V. Equations (5.17)–(5.19) with the energy conservation law (5.16) coincide with (5.1)–(5.3) and (5.6) respectively up to notation.

We also note that system (5.12)–(5.15) is written in the canonical form (1.1), i.e., the following [30] holds:

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial L^\alpha}{\partial q^\beta}\right)=F_\beta, \qquad \alpha=0,1,2, \qquad \beta=0,1,\dots,4, \tag{5.20}$$

$$L^0\equiv P, \quad L^k\equiv PV^k, \quad q^0\equiv gh-\frac{2}{3}W^2-\frac{V^2}{2}, \quad q^1\equiv V^1, \quad q^2\equiv V^2, \quad q^3\equiv\frac{4}{3}W, \quad q^4=\pi.$$

The vector $F=(F_\beta)$ on the right-hand side of system (5.20) is independent of derivatives and depends only on the unknown functions as which one can take (q^β)

$$F\equiv\left(0,0,0,\frac{3}{2h}\pi,-\frac{2}{h}W\right).$$

The form (5.20) enables us to write the rule of substitution of boundary conditions for the system (5.12)–(5.15) (see [7]).

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